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Zeros of orthogonal polynomials in certain discrete Sobolev spaces

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Abstract

In this work *discrete* Sobolev (pseudo-)inner products of type

$$\phi_1(p,q) := \lambda p(c)q(c) + \int_a^b p'(x)q'(x)\,d\mu(x),$$

where $d\mu$ is a quasi-definite Borel measure and $\lambda \neq 0$, and

$$\phi_2(p,q) := \lambda(p(c)q(c) + p(-c)q(-c)) + \int_{-a}^{a} p'(x)q'(x)\,d\mu(x),$$

where $d\mu$ is a symmetric positive Borel measure, $c \neq 0$, and $\lambda > 0$ are considered. General properties of orthogonal polynomials associated with the above *discrete* Sobolev inner products and their zeros are studied.

Keywords: Sobolev orthogonal polynomials; Discrete Sobolev inner products; Zeros

1. Introduction

Throughout the work, the term *Borel measure* always means a signed Borel measure $d\mu$ on an interval $I = (a, b)$, finite or infinite, of which the moments

$$\mu_n := \int_a^b x^n\,d\mu(x), \quad n = 0,1,2,\ldots,$$

are all finite. A Borel measure $d\mu$ is called *quasi-definite* (resp. *positive-definite*) if its moments $\{\mu_n\}_{n=0}^{\infty}$ satisfy

$$\Delta_n(d\mu) := \det[\mu_{i+j}]_{i,j=0}^{n} \neq 0 \text{ (resp. } \Delta_n(d\mu) > 0), \quad n \geqslant 0.$$

We call a Borel measure $d\mu$ *symmetric* if $\mu_{2n+1} = 0$, $n \geqslant 0$.

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We are concerned with the behavior of zeros of Sobolev orthogonal polynomials which are orthogonal relative to a Sobolev pseudo-inner product of type

$$\phi(p, q) := \int_I p(x)q(x)\,\mathrm{d}\sigma(x) + \int_{I'} p'(x)q'(x)\,\mathrm{d}\mu(x), \tag{1.1}$$

where $\mathrm{d}\sigma$ and $\mathrm{d}\mu(\neq 0)$ are Borel measures on intervals I and I', respectively. Many results on the zeros of such Sobolev orthogonal polynomials are already known for some special choices of $\mathrm{d}\sigma$ and $\mathrm{d}\mu$. For example, zeros of Sobolev orthogonal polynomials associated with *discrete* Sobolev inner products of type

$$\int_a^b p(x)q(x)\,\mathrm{d}\sigma(x) + Mp(c)q(c) + Np'(c)q'(c), \tag{1.2}$$

where $\mathrm{d}\sigma$ is a positive Borel measure, $M \geqslant 0$, and $N > 0$, are studied in [1, 2, 4, 10, 11]. Also, zeros of Sobolev orthogonal polynomials associated with *continuous* Sobolev inner products of type

$$\int_a^b p(x)q(x)\,\mathrm{d}\sigma(x) + \lambda\int_a^b p'(x)q'(x)\,\mathrm{d}\mu(x), \tag{1.3}$$

which $\mathrm{d}\sigma$ and $\mathrm{d}\mu$ are positive Borel measures and $\lambda \geqslant 0$, are studied in [3, 5, 7] for some special choices of $\mathrm{d}\sigma = \mathrm{d}\mu$. Recently, Meijer [12] obtained general results on the zeros of Sobolev orthogonal polynomials associated with (1.3) assuming only that the pair $\{\mathrm{d}\sigma, \mathrm{d}\mu\}$ is a coherent pair of measures (see Definition A in Section 3) or a symmetrically coherent pair of measures (see Definition B in Section 4).

The aim of this paper is to study the behavior of zeros of Sobolev orthogonal polynomials associated with *discrete* Sobolev (pseudo-)inner products of type

$$\phi_1(p, q) := \lambda p(c)q(c) + \int_a^b p'(x)q'(x)\,\mathrm{d}\mu(x), \tag{1.4}$$

where $\mathrm{d}\mu$ is a quasi-definite Borel measure and $\lambda \neq 0$, and

$$\phi_2(p, q) := \lambda(p(c)q(c) + p(-c)q(-c)) + \int_{-a}^a p'(x)q'(x)\,\mathrm{d}\mu(x), \tag{1.5}$$

where $\mathrm{d}\mu$ is a symmetric positive Borel measure, $c \neq 0$, and $\lambda > 0$.

As in [12], we use extensively the concept of coherent pair of measures but in a different context. Sobolev orthogonal polynomials associated with *discrete* Sobolev (pseudo-)inner products of type (1.4) or (1.5) arise naturally in classifying all polynomial solutions of second-order differential equation

$$L_2[y](x) = l_2(x)y''(x) + l_1(x)y'(x) = \lambda_n y(x),$$

which are orthogonal relative to Sobolev pseudo-inner products of type (1.1) (see [9]). For example, Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ and Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ for $\alpha = -1$ or $\beta = -1$ are not orthogonal in the *ordinary* sense but they have Sobolev orthogonality relative to *discrete* Sobolev (pseudo-)inner products of type (1.4) or (1.5).

In Section 2, we discuss some background results and notations that we shall need for the rest of paper. In Sections 3 and 4, we discuss zeros of Sobolev orthogonal polynomials associated with (1.4) and (1.5), respectively, and finally in Section 5, we give several examples.

2. Preliminaries

All polynomials in this work are assumed to be real polynomials of a real variable x and we let $\mathscr{P}$ be the space of all these polynomials. We use $\deg(\pi)$ to denote the degree of a polynomial $\pi(x)$ with the convention that $\deg(0) = -1$. By a polynomial system (PS), we mean a sequence of polynomials $\{\psi_n(x)\}_{n=0}^{\infty}$ with $\deg(\psi_n) = n$, $n \geqslant 0$.

For any symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathscr{P} \times \mathscr{P}$ we call the double sequence

$$\phi_{m,n} := \phi(x^m, x^n) \quad (m \text{ and } n \geqslant 0),$$

the moments of $\phi(\cdot,\cdot)$ and say that $\phi(\cdot,\cdot)$ is *quasi-definite* (resp. *positive-definite*) if its moments $\{\phi_{m,n}\}_{m,n=0}^{\infty}$ satisfy

$$\Delta_n(\phi) := \det[\phi_{i,j}]_{i,j=0}^{n} \neq 0 \text{ (resp. } \Delta_n(\phi) > 0), \quad n \geqslant 0. \tag{2.1}$$

In particular, when $\phi(\cdot,\cdot)$ is given by

$$\phi(p, q) = \int_a^b p(x)q(x)\,\mathrm{d}\mu(x) \tag{2.2}$$

for some Borel measure $\mathrm{d}\mu$ on (a, b), these notions reduce to those for Borel measures given in Section 1.

A symmetric bilinear form $\phi(\cdot,\cdot)$ on $\mathscr{P} \times \mathscr{P}$ is quasi-definite (resp. positive-definite) if and only if there is a PS $\{\psi_n(x)\}_{n=0}^{\infty}$ such that

$$\phi(\psi_m, \psi_n) = a_n\delta_{mn}, \quad m \text{ and } n \geqslant 0, \tag{2.3}$$

for some constants $a_n \neq 0$ (resp. $a_n > 0$). Moreover, in this case, each $\psi_n(x)$ is determined uniquely up to a nonzero constant multiple (see [9, Lemma 2.1]). In fact, we may take $\psi_n(x)$ to be

$$\psi_n(x) := \begin{vmatrix} \phi(1,1) & \phi(1, x-c) & \cdots & \phi(1, (x-c)^n) \\ \phi(x-c, 1) & \phi(x-c, x-c) & \cdots & \phi(x-c, (x-c)^n) \\ \vdots & \vdots & \ddots & \vdots \\ \phi((x-c)^{n-1}, 1) & \phi((x-c)^{n-1}, x-c) & \cdots & \phi((x-c)^{n-1}, (x-c)^n) \\ 1 & (x-c) & \cdots & (x-c)^n \end{vmatrix}, \tag{2.4}$$

where $n \geqslant 0$ and c is any constant.

When $\phi(\cdot,\cdot)$ is given by (2.2), we call a PS$\{\psi_n(x)\}_{n=0}^{\infty}$ satisfying (2.3) a Tchebychev polynomial system (TPS) or an orthogonal polynomial system (OPS) relative to $\mathrm{d}\mu$, depending whether $\mathrm{d}\mu$ is quasi-definite or positive-definite.

Lemma 2.1. *Let $\phi(\cdot,\cdot)$ be a symmetric bilinear form on $\mathscr{P}\times\mathscr{P}$. Then for any monic PS $\{\psi_n(x)\}_{n=0}^{\infty}$*

$$\Delta_n(\phi)=\det[\phi(\psi_i,\psi_j)]_{i,j=0}^{n},\quad n\geqslant 0. \tag{2.5}$$

Proof. For any integer $n\geqslant 0$, let $A=[a_{ij}]_{i,j=0}^{n}$ be an $(n+1)\times(n+1)$ matrix and set

$$\tilde{\psi}_j(x)=\sum_{k=0}^{n}a_{jk}\psi_k(x),\quad 0\leqslant j\leqslant n.$$

Then we have

$$A[\phi(\psi_i,\psi_j)]_{i,j=0}^{n}A^{t}=[\phi(\tilde{\psi}_i,\tilde{\psi}_j)]_{i,j=0}^{n}.$$

Therefore, we have (2.5) if we choose A to be a lower triangular matrix such that $a_{jj}=1$ and $\tilde{\psi}_j(x)=x^j$, $j=0,1,\dots,n$. □

Proposition 2.2. *The symmetric bilinear forms $\phi_1(\cdot,\cdot)$ in (1.4) and $\phi_2(\cdot,\cdot)$ in (1.5) are quasi-definite and positive-definite, respectively.*

Proof. For any $\pi(x)$, not identically zero, in $\mathscr{P}$, $\phi_2(\pi,\pi)>0$ so that $\phi_2(\cdot,\cdot)$ is positive-definite. For $\phi_1(\cdot,\cdot)$, we have

$$\phi_1((x-c)^m,(x-c)^n)=\begin{cases}\lambda & \text{if } m=n=0,\\ 0 & \text{if } m=0,\ n\geqslant 1 \text{ or } m\geqslant 1,\ n=0,\\ mn\,\tau_{m+n-2} & \text{if } m\geqslant 1 \text{ and } n\geqslant 1,\end{cases}$$

where $\tau_n=\int_a^b(x-c)^n\,d\mu$, $n\geqslant 0$. Then we have by Lemma 2.1

$$\Delta_n(\phi_1)=\lambda(n!)^2\Delta_{n-1}(d\mu)\neq 0,\quad n\geqslant 0\ \ (\Delta_{-1}(d\mu):=1)$$

so that $\phi_1(\cdot,\cdot)$ is quasi-definite. □

From now on, we let $\{P_n(x)\}_{n=0}^{\infty}$ be a TPS (resp. an OPS) relative to the Borel measure $d\mu$ in (1.4) (resp. in (1.5)) and let $\{Q_n(x)\}_{n=0}^{\infty}$ and $\{R_n(x)\}_{n=0}^{\infty}$ be Sobolev orthogonal polynomials relative to $\phi_1(\cdot,\cdot)$ in (1.4) and $\phi_2(\cdot,\cdot)$ in (1.5), respectively.

Proposition 2.3. *Sobolev orthogonal polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ satisfy*

(i) $Q_n(c)=0$, $n\geqslant 1$;

(ii) $Q_n'(x)=A_nP_{n-1}(x)$, $n\geqslant 1$, *for some constants* $A_n\neq 0$.

Proof. (i) It is trivial from the formula (2.4) since $\phi_1(1,(x-c)^k)=0$, $1\leqslant k\leqslant n$. (ii) For $0\leqslant k\leqslant n$, we have from (i)

$$\phi_1(Q_{n+1}(x),x^{k+1})=(k+1)\int_a^b Q_{n+1}'(x)x^k\,d\mu(x),$$

which is 0 only for $k<n$. Hence $\{Q_n'(x)\}_{n=1}^{\infty}$ is a TPS relative to $d\mu$ so that (ii) follows. □

Proposition 2.4. *Sobolev orthogonal polynomials* $\{R_n(x)\}_{n=0}^{\infty}$ *satisfy*

(i) $R_n(-x) = (-1)^n R_n(x)$, $n \geqslant 0$;

(ii) $R_n(-c) = -R_n(c)$, $n \geqslant 1$ *so that* $R_{2n}(c) = R_{2n}(-c) = 0$, $n \geqslant 1$;

(iii) $\int_{-a}^{a} R_n'(x) x^{2k-1}\, d\mu(x) = 0$, $n \geqslant 2$ *and* $k = 1, 2, \ldots, [n/2]$;

(iv) $R_{2n}'(x) = B_{2n} P_{2n-1}$, $n \geqslant 1$, *for some constants* $B_{2n} \neq 0$.

Proof. (i) Since $d\mu$ in (1.5) is symmetric, we have

$$\int_{-a}^{a} \pi(x)\, d\mu(x) = \int_{-a}^{a} \pi(-x)\, d\mu(x)$$

for any $\pi(x)$ in $\mathscr{P}$ so that

$$\phi_2(p(-x), q(-x)) = \phi_2(p(x), q(x))$$

for any $p(x)$ and $q(x)$ in $\mathscr{P}$. In particular, we have

$$\phi_2(R_m(-x), R_n(-x)) = \phi_2(R_m(x), R_n(x)), \quad m \text{ and } n \geqslant 0$$

so that $\{R_n(-x)\}_{n=0}^{\infty}$ is also orthogonal relative to $\phi_2(\cdot,\cdot)$. Hence, $R_n(-x) = C_n R_n(x)$ for some constants $C_n \neq 0$, $n \geqslant 0$. By comparing the coefficients of x^n from both sides, we have $C_n = (-1)^n$, $n \geqslant 0$.

(ii) We have from the orthogonality

$$0 = \phi_2(R_n(x), 1) = \lambda(R_n(c) + R_n(-c)), \quad n \geqslant 1$$

so that $R_n(-c) = -R_n(c)$, $n \geqslant 1$. It implies $R_{2n}(c) = R_{2n}(-c) = 0$, $n \geqslant 1$, by (i).

(iii) We have from (ii) and the orthogonality

$$\begin{aligned} 0 &= \phi_2(R_n(x), x^k) \\ &= \lambda R_n(c) c^k (1 + (-1)^{k+1}) + k \int_{-a}^{a} R_n'(x) x^{k-1}\, d\mu(x), \quad 0 \leqslant k < n. \end{aligned} \tag{2.6}$$

Hence, $\int_{-a}^{a} R_n'(x) x^{k-1}\, d\mu(x) = 0$ if k is an even integer with $0 \leqslant k < n$.

(iv) We have from (ii) and the orthogonality

$$\begin{aligned} 0 &= \phi_2(R_{2n}(x), x^{k+1}) \\ &= (k+1) \int_{-a}^{a} R_{2n}'(x) x^k\, d\mu(x), \quad 0 \leqslant k \leqslant 2n-2 \text{ and } n \geqslant 1. \end{aligned} \tag{2.7}$$

If we write $R_{2n}'(x) = \sum_{j=0}^{2n-1} C_j P_j(x)$ with $C_{2n-1} \neq 0$, then we have from (2.7)

$$\begin{aligned} 0 &= \int_{-a}^{a} R_{2n}'(x) P_k(x)\, d\mu(x) = \sum_{j=0}^{2n-1} C_j \int_{-a}^{a} P_j(x) P_k(x)\, d\mu(x) \\ &= C_k \int_{-a}^{a} P_k^2(x)\, d\mu(x), \quad 0 \leqslant k \leqslant 2n-2, \end{aligned}$$

so that $C_k = 0$, $0 \leqslant k \leqslant 2n-2$. $\square$

Remark 2.5. From Proposition 2.4 (ii) and the Eq. (2.6), we can see that for any integer $n \geqslant 2$, $R_n(c) = R_n(-c) = 0$ if and only if $R_n'(x) = B_n P_{n-1}(x)$ for some constant $B_n \neq 0$.

If we set

$$b_{m,n} := mn \int_{-a}^{a} (x+c)^{m+n-2}\, \mathrm{d}\mu(x) \quad (m \text{ and } n \geqslant 1),$$

then the matrix $[b_{i,j}]_{i,j=1}^{n}$ is positive-definite for every $n \geqslant 1$ and each $R_n(x)$ can be given (up to a nonzero constant multiple) by

$$R_n(x) := \begin{vmatrix} 2\lambda & \lambda(2c) & \cdots & \lambda(2c)^n \\ \lambda(2c) & \lambda(2c)^2 + b_{1,1} & \cdots & \lambda(2c)^{n+1} + b_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda(2c)^{n-1} & \lambda(2c)^n + b_{n-1,1} & \cdots & \lambda(2c)^{2n-1} + b_{n-1,n} \\ 1 & (x+c) & \cdots & (x+c)^n \end{vmatrix}, \quad n \geqslant 0 \tag{2.8}$$

Therefore, $R_1(c) = 2c\lambda \neq 0$. For $n \geqslant 2$, we have the following.

Proposition 2.6. *For any integer $l \geqslant 1$, the following statements are all equivalent.*

(i) $R_n(-c) = R_n(c) = 0$, $n \geqslant l+1$.

(ii) *There are l constants $\{a_j\}_{j=1}^{l}$, not all zero, such that*

$$\frac{a_1 b_{1,k} + a_2 b_{2,k} + \cdots + a_l b_{l,k}}{a_1 b_{1,1} + a_2 b_{2,1} + \cdots + a_l b_{l,1}} = (2c)^{k-1} \quad \textit{for } k \geqslant 1. \tag{2.9}$$

(iii) *There are constants $B_n \neq 0$ such that $R_n'(x) = B_n P_{n-1}(x)$, $n \geqslant l+1$.*

Proof. The statements (i) and (iii) are equivalent by Remark 2.5. For $n \geqslant 2$, we have from (2.8)

$$R_n(-c) = \lambda \begin{vmatrix} (2c) & (2c)^2 & \cdots & (2c)^n \\ b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n-1,1} & b_{n-1,2} & \cdots & b_{n-1,n} \end{vmatrix}. \tag{2.10}$$

Assume that the condition (ii) holds. Then we have

$$A \sum_{j=1}^{l} a_j b_{j,k} = (2c)^k, \quad k \geqslant 1,$$

where $A = 2c \sum_{j=1}^{l} a_j b_{j,1}$ so that the n row vectors in (2.10) are linearly dependent for $n \geqslant l+1$. Hence, by Proposition 2.4(ii), $R_n(-c) = R_n(c) = 0$ for $n \geqslant l+1$. Conversely assume $R_n(-c) = R_n(c) = 0$ for $n \geqslant l+1$. Then, for $n = l+1$ in (2.10), the $l+1$ row vectors in (2.10) are linearly dependent so that there are l constants $\{a_j\}_{j=1}^{l}$, not all zero, such that

$$\sum_{j=1}^{l} a_j b_{j,k} = (2c)^k, \quad 1 \leqslant k \leqslant l+1, \tag{2.11}$$

since the last l row vectors in (2.10) are linearly independent. Similarly for $n = l + 2$ in (2.10) there are $l + 1$ constants $\{c_j\}_{j=1}^{l+1}$, not all zero, such that

$$\sum_{j=1}^{l+1} c_j b_{j,k} = (2c)^k, \quad 1 \leqslant k \leqslant l + 2. \tag{2.12}$$

Subtracting (2.11) from (2.12), we thus obtain

$$\sum_{j=1}^{l} (c_j - a_j) b_{j,k} + c_{l+1} b_{l+1,k} = 0, \quad 1 \leqslant k \leqslant l + 1,$$

which implies $c_j = a_j$, $1 \leqslant j \leqslant l$ and $c_{l+1} = 0$ since $\det[b_{i,j}]_{i,j=1}^{l+1} \neq 0$. Therefore, (2.11) holds for $1 \leqslant k \leqslant l + 2$. Continuing the same process, we have

$$\sum_{j=1}^{l} a_j b_{j,k} = (2c)^k, \quad k \geqslant 1,$$

which implies (2.9). □

Corollary 2.7. *If there is an integer $l \geqslant 1$ such that*

$$\frac{b_{l,k}}{b_{l,l}} = (2c)^{k-1}, \quad k \geqslant 1, \tag{2.13}$$

then $R_n(c) = R_n(-c) = 0$, $n \geqslant l + 1$.

Proof. It is enough to take $a_1 = a_2 = \cdots = a_{l-1} = 0$ and $a_l = 1$ in Proposition 2.6. □

3. Zeros of $Q_n(x)$

We now investigate the behavior of zeros of Sobolev orthogonal polynomials $\{Q_n(x)\}_{n=0}^{\infty}$ relative to the quasi-definite bilinear form $\phi_1(\cdot,\cdot)$ in (1.4). As in Section 2, we let $\{P_n(x)\}_{n=0}^{\infty}$ be a TPS relative to $\mathrm{d}\mu$ in (1.4).

Lemma 3.1. *Let $\{S_n(x)\}_{n=0}^{\infty}$ be an OPS relative to a positive-definite Borel measure $\mathrm{d}\tau$ on (a, b) and $\pi(x)$ a polynomial of degree $m \geqslant 1$.*

(i) *If $\pi(x) = \sum_{j=k}^{m} C_j S_j(x)$, $C_m \neq 0$, and $k \geqslant 0$ is an integer, then $\pi(x)$ has at least k zeros of odd multiplicity in (a, b).*

(ii) *If $\pi(x) = C_m S_m(x) + C_{m-1} S_{m-1}(x)$, $C_m \neq 0$, then $\pi(x)$ has m simple real zeros which interlace with the zeros of $S_{m-1}(x)$ and at least $m - 1$ zeros of $\pi(x)$ lie in (a, b).*

Proof. We first recall that for $n \geqslant 1$, $S_n(x)$ has n simple real zeros in (a, b) which interlace with the zeros of $S_{n+1}(x)$ (see [6, Ch. 1]). (i) Let $x_1 < x_2 < \cdots < x_l$ be the zeros of $\pi(x)$ of odd multiplicity in (a, b) and $h(x) = \prod_{i=1}^{l}(x - x_i)$. Then we may assume $h(x)\pi(x) \geqslant 0$ on (a, b) so that

$$\int_a^b h(x)\pi(x)\,\mathrm{d}x = \sum_{j=k}^{m} C_j \int_a^b h(x) S_j(x)\,\mathrm{d}x > 0.$$

Therefore, we must have $\deg(h) = l \geqslant k$. (ii) By (i), $\pi(x)$ has at least $m-1$ zeros of odd multiplicity in (a,b). Hence, $\pi(x)$ has m simple real zeros. If $\{y_j\}_{j=1}^{m-1}$ are the zeros of $S_{m-1}(x)$, then $\pi(y_j) = C_m S_m(y_j)$, $1 \leqslant j \leqslant m-1$. Therefore, the zeros of $\pi(x)$ interlace with the zeros of S_{m-1}. □

Definition A (Iserles et al. [8]). We call a pair $\{\mathrm{d}\sigma, \mathrm{d}\tau\}$ of quasi-definite Borel measures a *coherent pair* if there are nonzero constants C_n and D_n such that

$$H_n(x) = C_n K'_{n+1}(x) - D_n K'_n(x), \quad n \geqslant 1, \tag{3.1}$$

where $\{K_n(x)\}_{n=0}^{\infty}$ and $\{H_n(x)\}_{n=0}^{\infty}$ are TPSs relative to $\mathrm{d}\sigma$ and $\mathrm{d}\tau$, respectively.

Theorem 3.2. *Let $m \geqslant 2$ be an integer. If there is a positive-definite Borel measure* $\mathrm{d}\sigma$ *on* (a,b) *such that*
(a) $\int_a^b Q_m(x)\,\mathrm{d}\sigma(x) = 0$;
(b) $\{\mathrm{d}\sigma, \mathrm{d}\mu\}$ *is a coherent pair,*
then $Q_m(x)$ has m simple real zeros which interlace with the zeros of $K_{m-1}(x)$ and at least $m-1$ zeros of $Q_m(x)$ lie in (a,b). If, furthermore, $\mathrm{d}\mu$ is positive-definite, then the zeros of $Q_m(x)$ also interlace with the zeros of $P_{m-1}(x)$. Here, $\{K_n(x)\}_{n=0}^{\infty}$ is an OPS relative to $\mathrm{d}\sigma$.

Proof. By definition of coherence, we have

$$P_n(x) = C_n K'_{n+1}(x) - D_n K'_n(x), \quad (C_n \neq 0 \text{ and } D_n \neq 0), \quad n \geqslant 1.$$

Then by Proposition 2.3(ii) we have

$$Q'_n(x) = A_n(C_{n-1}K'_n(x) - D_{n-1}K'_{n-1}(x)), \quad n \geqslant 2$$

and so

$$Q_n(x) = A_n(C_{n-1}K_n(x) - D_{n-1}K_{n-1}(x)) + E_n, \quad n \geqslant 2 \tag{3.2}$$

for some constant E_n. Integrating (3.2) for $n = m$ with respect to $\mathrm{d}\sigma$, we have $E_m = 0$ from (a) so that Eq. (3.2) for $n = m$ becomes

$$Q_m(x) = A_m(C_{m-1}K_m(x) - D_{m-1}K_{m-1}(x)). \tag{3.3}$$

Hence, the first conclusion follows from Lemma 3.1(ii). Now we assume that $\mathrm{d}\mu$ is positive-definite. Then $P_{m-1}(x)$ has $m-1$ simple real zeros $\{x_j\}_{j=1}^{m-1}$. Since $Q'_m(x) = A_m P_{m-1}(x)$, $A_m \neq 0$, by Proposition 2.3(ii), $Q_m(x)$ is monotone on each are of m intervals $(-\infty, x_1), (x_1, x_2), \ldots, (x_{m-1}, \infty)$. Since all zeros of $Q_m(x)$ are simple, $Q_m(x_j) \neq 0$, $1 \leqslant j \leqslant m-1$. Therefore, $Q_m(x)$ must have exactly one zero in each one of m intervals $(-\infty, x_1), (x_1, x_2), \ldots, (x_{m-1}, \infty)$. □

We can generalize Theorem 3.2 as follows.

Theorem 3.3. *Let $m \geqslant 2$ be an integer. If there are positive-definite Borel measure* $\mathrm{d}\sigma$ *on* (a,b) *and polynomial $\pi(x)$ of degree r, $0 \leqslant r \leqslant m-2$, such that*
(a) $\int_a^b Q_m(x)\,\mathrm{d}\sigma(x) = 0$;
(b) $\pi(x)\,\mathrm{d}\mu$ *is quasi-definite*;
(c) $\{\mathrm{d}\sigma, \pi(x)\,\mathrm{d}\mu\}$ *is a coherent pair,*
then $Q_m(x)$ has at least $m-r-1$ zeros of odd multiplicity in (a,b).

Proof. Let $\{K_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=0}^{\infty}$ be TPSs relative to $\mathrm{d}\sigma$ and $\pi(x)\,\mathrm{d}\mu$, respectively. Then

$$H_n(x) = C_n K'_{n+1}(x) - D_n K'_n(x), \quad (C_n \neq 0 \text{ and } D_n \neq 0), \quad n \geqslant 1. \tag{3.4}$$

Let us write $P_n(x)$ as

$$P_n(x) = \sum_{j=0}^{n} a_j^n H_j(x), \quad (a_n^n \neq 0), \quad n \geqslant 0. \tag{3.5}$$

Multiplying (3.5) by $H_k(x)\pi(x)$ and integrating with respect to $\mathrm{d}\mu$, we obtain

$$a_k^n \int_a^b H_k(x)^2 \pi(x)\,\mathrm{d}\mu(x) = \int_a^b P_n(x) H_k(x)\pi(x)\,\mathrm{d}\mu(x) = 0, \quad 0 \leqslant k \leqslant n-(r+1),$$

so that $a_k^n = 0, 0 \leqslant k \leqslant n-(r+1)$. Therefore, Eq. (3.5) reduces to

$$P_n(x) = \sum_{j=n-r}^{n} a_j^n H_j(x), \quad n \geqslant 0. \tag{3.6}$$

where $a_j^n = 0$ if $j < 0$. Then we have from Proposition 2.3(ii) and (3.4)

$$Q'_n(x) = A_n \sum_{j=n-r-1}^{n-1} a_j^{n-1}(C_j K'_{j+1}(x) - D_j K'_j(x))$$

and so

$$Q_n(x) = A_n \sum_{j=n-r-1}^{n-1} a_j^{n-1}(C_j K_{j+1}(x) - D_j K_j(x)) + E_n, \quad n \geqslant 1 \tag{3.7}$$

for some constant E_n. Integrating (3.7) for $n = m$ with respect to $\mathrm{d}\sigma$, we have $E_m = 0$ by (a) so that

$$Q_m(x) = \sum_{j=m-r-1}^{m} b_j^m K_j(x). \tag{3.8}$$

Therefore, the conclusion follows from Lemma 3.1 (i). □

4. Zeros of $R_n(x)$

We now investigate the behavior of zeros of Sobolev orthogonal polynomials $\{R_n(x)\}_{n=0}^{\infty}$ relative to the positive-definite bilinear form $\phi_2(\cdot,\cdot)$ in (1.5). In this section we always assume that there is an integer $l = l(\phi_2) \geqslant 1$ for which the condition (2.9) holds so that $R'_n(x) = B_n P_{n-1}(x)$, $n \geqslant l+1$, where $\{P_n(x)\}_{n=0}^{\infty}$ is an OPS relative to $\mathrm{d}\mu$ (see Proposition 2.6).

Then by the same arguments as in Theorems 3.2 and 3.3, we can obtain the following two results on the zeros of $R_n(x)$.

Theorem 4.1. *Let $m \geqslant l+1$ be an integer. If there is a positive-definite Borel measure $\mathrm{d}\sigma$ on $(-a, a)$ such that*

(a) $\int_{-a}^{a} R_m(x)\,\mathrm{d}\sigma(x) = 0$;

(b) $\{\mathrm{d}\sigma, \mathrm{d}\mu\}$ *is a coherent pair,* *then $R_m(x)$ has m simple real zeros which interlace with the zeros of $K_{m-1}(x)$ and $P_{m-1}(x)$ and at least $m-1$ zeros of $R_m(x)$ lie in $(-a, a)$. Here, $\{K_n(x)\}_{n=0}^{\infty}$ is an OPS relative to* $d\sigma$.

Theorem 4.2. *Let $m \geqslant l+1$ be an integer. If there are positive-definite Borel measure* $d\sigma$ *on $(-a, a)$ and polynomial $\pi(x)$ of degree r, $0 \leqslant r \leqslant m-2$, such that*
(a) $\int_{-a}^{a} R_m(x)\, d\sigma(x) = 0$;
(b) $\pi(x)\, d\mu$ *is quasi-definite;*
(c) $\{d\sigma, \pi(x)\, d\mu\}$ *is a coherent pair,*
then $R_m(x)$ has at least $m-r-1$ zeros of odd multiplicity in $(-a, a)$.

In Theorems 4.1 and 4.2, the symmetry of the measure $d\mu$ in (1.5) plays no role. Below we give results on the zeros of $R_n(x)$ which are derivable from the symmetry of $d\mu$.

Definition B (Iserles [8]). We call a pair $\{d\sigma, d\tau\}$ of symmetric positive-definite Borel measures a *symmetrically coherent pair* if there are non-zero constants C_n and D_n such that

$$H_n(x) = C_n K'_{n+1}(x) - D_n K'_{n-1}(x), \quad n \geqslant 2, \tag{4.1}$$

where $\{K_n(x)\}_{n=0}^{\infty}$ and $\{H_n(x)\}_{n=0}^{\infty}$ are OPSs relative to $d\sigma$ and $d\tau$, respectively.

Note that any pair of symmetric Borel measures cannot be a coherent pair in the sense of Definition A in Section 3.

Lemma 4.3. *Let $m \geqslant \max(3, l+1)$ be an integer. If there is a symmetric positive-definite Borel measure* $d\sigma$ *on $(-a, a)$ such that*
(a) $\int_{-a}^{b} R_m(x)\, d\sigma(x) = 0$;
(b) $\{d\sigma, d\mu\}$ *is a symmetrically coherent pair,*
then $R_m(x)$ has at least $m-2$ simple zeros in $(-a, a)$.

Proof. By definition of symmetrical coherence, we have

$$P_n(x) = C_n K'_{n+1}(x) - D_n K'_{n-1}(x), \quad n \geqslant 2,$$

where $\{K_n(x)\}_{n=0}^{\infty}$ is an OPS relative to $d\sigma$. Then by Proposition 2.6(iii) we have

$$R'_n(x) = B_n(C_{n-1}K'_n(x) - D_{n-1}K'_{n-2}(x)), \quad n \geqslant \max(3, l+1)$$

and so

$$R_n(x) = B_n(C_{n-1}K_n(x) - D_{n-1}K_{n-2}(x)) + E_n, \quad n \geqslant \max(3, l+1) \tag{4.2}$$

for some constant E_n. Integrating (4.2) for $n = m$ with respect to $d\sigma$, we have $E_m = 0$ from (a) so that Eq. (4.2) for $n = m$ becomes

$$R_m(x) = B_m(C_{m-1}K_m(x) - D_{m-1}K_{m-2}(x)). \tag{4.3}$$

Therefore by Lemma 3.1(i), $R_m(x)$ has at least $m-2$ zeros $\{x_j\}_{j=0}^{m-2}$ of odd multiplicity in $(-a, a)$. If some x_j has multiplicity $\geqslant 3$, then $x = x_j$ is a zero of $P_{m-1}(x)$ of multiplicity $\geqslant 2$ since $R'_m(x) = B_m P_{m-1}(x)$. It is a contradiction. Hence all x_j, $1 \leqslant j \leqslant m-2$, must be simple. $\square$

In the following, we use the notation $-I$ to denote the interval $(-b, -a)$ for an interval $I = (a, b)$.

Theorem 4.4. *Let $m = 2k \geqslant \max(4, l+1)$ be an even integer. Assume that there is a symmetric positive-definite Borel measure $d\sigma$ on $(-a, a)$ satisfying the conditions* (a) *and* (b) *in Lemma* 4.3. *Let $\{P_n(x)\}_{n=0}^{\infty}$ and $\{K_n(x)\}_{n=0}^{\infty}$ be OPSs relative to* $d\mu$ *and* $d\sigma$, *respectively, and let $\{z_j\}_{j=1}^{k-1}$ and $\{w_j\}_{j=1}^{k-1}$ be positive zeros of $P_{2k-1}(x)$ and $K_{2k-2}(x)$, respectively. Then $R_{2k}(x)$ satisfies exactly one of the following.*

(i) *$R_{2k}(x)$ has exactly one zero in each one of $2k$ intervals $\{\pm I_j\}_{j=1}^{k}$ or $\{\pm J_j\}_{j=1}^{k}$.*

(ii) *$R_{2k}(x)$ has exactly one zero in each one of $2k-2$ intervals $\{\pm I_j\}_{j=2}^{k}$ or $\{\pm J_j\}_{j=2}^{k}$ and has $x = 0$ as a zero of multiplicity two.*

(iii) *$R_{2k}(x)$ has exactly one zero in each one of $2k-2$ intervals $\{\pm I_j\}_{j=2}^{k}$ or $\{\pm J_j\}_{j=2}^{k}$ and has two complex conjugate zeros.*

Here $I_j = (w_{j-1}, w_j)$, $J_j = (z_{j-1}, z_j)$, $1 \leqslant j \leqslant k$ and $w_0 = z_0 = 0$, $w_k = z_k = \infty$.

Proof. Define a PS $\{U_n\}_{n=0}^{\infty}$ by the relations $U_n(x^2) = K_{2n}(x)$, $n \geqslant 0$. Then $\{U_n\}_{n=0}^{\infty}$ is an OPS and each $U_n(x)$, $n \geqslant 1$, has n simple zeros in $(0, a^2)$ (see [6]). On the other hand, we have by (4.3)

$$R_{2k}(x) = \alpha K_{2k}(x) - \beta K_{2k-2}(x)(\alpha \neq 0 \text{ and } \beta \neq 0). \tag{4.4}$$

If we define a polynomial $H_k(x)$ by

$$H_k(x) = \alpha U_k(x) - \beta U_{k-1}(x), \tag{4.5}$$

then $R_{2k}(x) = H_k(x^2)$ and by Lemma 3.1(ii), $H_k(x)$ has k simple zeros $\{y_j\}_{j=1}^{k}$ satisfying

$$y_1 < x_1 < y_2 < \cdots < y_{k-1} < x_{k-1} < y_k,$$

where $\{x_j\}_{j=1}^{k-1}$ are zeros of $U_{k-1}(x)$.

Note also that zeros of $K_{2k-2}(x)$ are $\{\pm\sqrt{x_j}\}_{j=1}^{k-1}$. Since at least $k-1$ of $\{y_j\}_{j=1}^{k}$ lie in $(0, a^2)$, there arise three cases: $y_1 > 0$, $y_1 = 0$, and $y_1 < 0$. If $y_1 > 0$, then zeros of $R_{2k}(x)$ are $\{\pm\sqrt{y_j}\}_{j=1}^{k}$ so that the conclusion (i) for I_j follows. If $y_1 = 0$, then $y_1 = 0$ is a zero of multiplicity two for $R_{2k}(x)$ and all other zeros of $R_{2k}(x)$ are $\{\pm\sqrt{y_j}\}_{j=2}^{k}$. Hence the conclusion (ii) for I_j follows. If $y_1 < 0$, then $R_{2k}(x)$ has $2k-2$ real zeros $\{\pm\sqrt{y_j}\}_{j=2}^{k}$ and two complex conjugate zeros so that the conclusion (iii) for I_j follows.

Since $R'_{2k}(x) = B_{2k}P_{2k-1}(x)$ by Proposition 2.6(iii), $R_{2k}(x)$ is monotone on each one of $2k$ intervals $\{\pm J_j\}_{j=1}^{k}$ so that each $\pm J_j$ has at most one zero of $R_{2k}(x)$. If $R_{2k}(x)$ has no zero in J_j for some $j \geqslant 2$, then $R_{2k}(x)$ has no zero either in J_{j-1} or in J_{j+1}. Then, by symmetry, there are at least four such intervals which contain no zero of $R_{2k}(x)$. Then $R_{2k}(x)$ can have at most $2k-4$ simple real zeros since $\pm z_j$, $1 \leqslant j \leqslant k-1$, and 0 cannot be a simple zero of $R_{2k}(x)$. It contradicts Lemma 4.3. Therefore, $R_{2k}(x)$ must have exactly one zero in each one of $2k-2$ intervals $\{\pm J_j\}_{j=2}^{k}$. If $R_{2k}(z_j) = 0$ for some $1 \leqslant j \leqslant k-1$, then, by symmetry, $\pm z_j$ are zeros of $R_{2k}(x)$ of multiplicity $\geqslant 2$ since $R'_{2k}(\pm z_j) = 0$. Then $R_{2k}(x)$ can have at most $2k-4$ simple real zeros, which contradicts Lemma 4.3. Thus, we have the conclusion (i) or (ii) or (iii) corresponding to the above three cases. $\square$

Theorem 4.5. *Let $m = 2k + 1 \geqslant \max(3, l + 1)$ be an odd integer. Assume that there is a symmetric positive-definite Borel measure $\mathrm{d}\sigma$ on $(-a, a)$ satisfying the conditions* (a) *and* (b) *in Lemma* 4.3. *Then $R_{2k+1}(x)$ has $2k + 1$ simple real zeros which interlace with the zeros of $P_{2k}(x)$ and k positive (resp. k negative) zeros of $R_{2k+1}(x)$ interlace with the $k - 1$ positive (resp. $k - 1$ negative) zeros of $K_{2k-1}(x)$. Here, $\{P_n(x)\}_{n=0}^{\infty}$ and $\{K_n(x)\}_{n=0}^{\infty}$ are OPSs relative to $\mathrm{d}\mu$ and $\mathrm{d}\sigma$, respectively.*

Proof. Let $\{z_j\}_{j=1}^{k}$ be the positive zeros of $P_{2k}(x)$. Since $R'_{2k+1}(x) = B_{2k+1}P_{2k}(x)$ by Proposition 2.6(iii), $R_{2k+1}(x)$ is monotone on each one of $2k + 1$ intervals $J_0 = (-z_1, z_1)$ and $\{\pm J_j\}_{j=1}^{k}$, where $J_j = (z_j, z_{j+1})$, $1 \leqslant j \leqslant k$, and $z_{k+1} = \infty$. Hence, $R_{2k+1}(x)$ can have at most one zero in each one of $2k$ intervals $\{\pm J_j\}_{j=1}^{k}$. If $R_{2k+1}(x)$ has no zero in J_j for some $j \geqslant 1$, then $R_{2k+1}(x)$ has no zero either in J_{j-1} or in J_{j+1}. Then, by symmetry, there are at least four such intervals which contain no zero of $R_{2k+1}(x)$. Then $R_{2k+1}(x)$ can have at most $2k - 3$ simple real zeros since $\pm z_j$, $1 \leqslant j \leqslant k$, cannot be a simple zero of $R_{2k+1}(x)$. It contradicts Lemma 4.3. Therefore $R_{2k+1}(x)$ has exactly one zero in each one of $2k + 1$ intervals J_0 and $\{\pm J_j\}_{j=1}^{k}$ since $R_{2k+1}(0) = 0$. Hence, zeros of $R_{2k+1}(x)$ interlace with zeros of $P_{2k}(x)$.

Define a PS$\{V_n\}_{n=0}^{\infty}$ by the relations $xV_n(x^2) = K_{2n+1}(x)$, $n \geqslant 0$. Then $\{V_n\}_{n=0}^{\infty}$ is an OPS and each $V_n(x)$, $n \geqslant 1$, has n simple zeros in $(0, a^2)$ (see [6]). On the other hand, we have by (4.3)

$$R_{2k+1}(x) = \alpha K_{2k+1}(x) - \beta K_{2k-1}(x) (\alpha \neq 0 \text{ and } \beta \neq 0). \tag{4.6}$$

If we define a polynomial $G_k(x)$ by

$$G_k(x) = \alpha V_k(x) - \beta V_{k-1}(x), \tag{4.7}$$

then $R_{2k+1}(x) = xG_k(x^2)$ and by Lemma 3.1(ii), $G_k(x)$ has k simple zeros $\{y_j\}_{j=1}^{k}$ satisfying

$$y_1 < x_1 < y_2 < \cdots < y_{k-1} < x_{k-1} < y_k,$$

where $\{x_j\}_{j=1}^{k-1}$ are zeros of $V_{k-1}(x)$.

Note also that zeros of $K_{2k-1}(x)$ are $\{\pm\sqrt{x_j}\}_{j=2}^{k-1} \cup \{0\}$. If $y_1 = 0$, then $y_1 = 0$ is a zero of $R_{2k+1}(x)$ of multiplicity $\geqslant 3$. It contradicts the fact that $R_{2k+1}(x)$ has $2k + 1$ simple zeros. If $y_1 < 0$, then $R_{2k+1}(x)$ has complex zeros, which contradicts the fact that $R_{2k+1}(x)$ has only real zeros. Therefore, $y_1 > 0$ so that k positive (resp., k negative) zeros of $R_{2k+1}(x)$ interlace with the $k - 1$ positive (resp. $k - 1$ negative) zeros of $K_{2k-1}(x)$. □

By the same arguments as in Theorem 3.3, Lemma 4.3 can be extended as follows.

Theorem 4.6. *Let $m \geqslant \max(3, l + 1)$ be an integer. If there are symmetric positive-definite Borel measure $\mathrm{d}\sigma$ on $(-a, a)$ and polynomial $\pi(x)$ of degree r, $0 \leqslant r \leqslant m - 2$, such that*

(a) $\int_{-a}^{a} R_m(x)\,\mathrm{d}\sigma(x) = 0$;

(b) *$\pi(x)\,\mathrm{d}\mu$ is symmetric and quasi-definite;*

(c) *$\{\mathrm{d}\sigma, \pi(x)\,\mathrm{d}\mu\}$ is a symmetrically coherent pair,*

then $R_m(x)$ has at least $m - r - 2$ zeros of odd multiplicity in $(-a, a)$.

5. Examples

Consider the Laguerre differential equation

$$L[y](x) = xy''(x) + (\alpha + 1 - x)y'(x) = -ny(x). \tag{5.1}$$

For each $n \geqslant 0$, Eq. (5.1) has a unique (up to a nonzero constant multiple) polynomial solution of degree n, called the Laguerre polynomials;

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad n \geqslant 0. \tag{5.2}$$

Here, we use the notation

$$\binom{a}{0} = 1 \text{ and } \binom{a}{k} = \frac{a(a-1)\dots(a-k+1)}{k!}$$

for any real number a and any integer $k \geqslant 1$. It is well known that the Laguerre PS$\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ is a TPS (resp. an OPS) if and only if $\alpha \notin \{-1, -2, \dots\}$ (resp. $\alpha > -1$).

Example 1. Laguerre polynomials $\{L_n^{(-1)}(x)\}_{n=0}^{\infty}$.

It is shown in [9] that $\{L_n^{(-1)}(x)\}_{n=0}^{\infty}$ has Sobolev orthogonality relative to

$$(p, q)_\lambda := \lambda p(0)q(0) + \int_0^\infty p'(x)q'(x)\mathrm{e}^{-x}\,\mathrm{d}x, \quad \lambda \neq 0, \tag{5.3}$$

which is quasi-definite for $\lambda \neq 0$ and positive-definite for $\lambda > 0$. We have from Proposition 2.3 and (5.2)

$$L_n^{(-1)}(0) = 0, \quad n \geqslant 1; \tag{5.4}$$

$$L_n^{(-1)}(x)' = -L_{n-1}^{(0)}(x), \quad n \geqslant 1. \tag{5.5}$$

Then we have by integration by parts

$$\int_0^\infty L_n^{(-1)}(x)\mathrm{e}^{-x}\,\mathrm{d}x = -\int_0^\infty L_{n-1}^{(0)}(x)\mathrm{e}^{-x}\,\mathrm{d}x = 0, \quad n \geqslant 2$$

since $\{L_n^{(0)}(x)\}_{n=0}^{\infty}$ is an OPS relative to $\mathrm{e}^{-x}\,\mathrm{d}x$ *on* $(0, \infty)$. On the other hand, $\{\mathrm{e}^{-x}\,\mathrm{d}x, \mathrm{e}^{-x}\,\mathrm{d}x\}$ is a coherent pair of measures on $(0, \infty)$ since we have (see [14])

$$L_n^{(0)}(x) = L_n^{(0)}(x)' - L_{n+1}^{(0)}(x)', \quad n \geqslant 1.$$

Therefore, by Theorem 3.2 and (5.4), $L_n^{(-1)}(x)$, $n \geqslant 1$, has n simple zeros in $[0, \infty)$ which interlace with the zeros of $L_{n-1}^{(0)}(x)$. We also have (see [14])

$$L_n^{(-1)}(x) = \frac{-x}{n} L_{n-1}^{(1)}(x), \quad n \geqslant 1,$$

so that the zeros of $L_n^{(1)}(x)$ interlace with the zeros of $L_n^{(0)}(x)$, for $n \geqslant 1$.

We now consider the Jacobi differential equation

$$L[y](x) = (1 - x^2)y''(x) + [(\beta - \alpha) - (\alpha + \beta + 2)x]y'(x) = -n(n + \alpha + \beta + 1)y(x). \tag{5.6}$$

When $\alpha+\beta+1\notin\{-1,-2,\ldots\}$, Eq. (5.6) has a unique (up to a nonzero constant multiple) polynomial solution of degree n, for each $n\geqslant 0$, called the Jacobi polynomials;

$$P_n^{(\alpha,\beta)}(x)=2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}(x-1)^k(x+1)^{n-k},\quad n\geqslant 0. \tag{5.7}$$

It is well known that the Jacobi PS $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ is a TPS (resp. an OPS) if and only if $\alpha+\beta+1$, α, $\beta\notin\{-1,-2,\ldots\}$ (resp. α and $\beta>-1$). It is shown in [9] that $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ has Sobolev orthogonality in case $\alpha=-1$, $\beta\notin\{-1,-2,\ldots\}$ or $\beta=-1$, $\alpha\notin\{-1,-2,\ldots\}$.

Example 2. Jacobi polynomials $\{P_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty}$ for $\alpha>-1$.

For $\alpha>-1$, the Jacobi PS $\{P_n^{(\alpha,-1)}(x)\}_{n=\infty}^{\infty}$ is orthogonal relative to

$$(p,q)_\lambda^\alpha:=\lambda p(-1)q(-1)+\int_{-1}^{1}p'(x)q'(x)(1-x)^{\alpha+1}\,\mathrm{d}x,\quad \lambda\neq 0, \tag{5.8}$$

which is quasi-definite for $\lambda\neq 0$ and positive-definite for $\lambda>0$. We have from Proposition 2.3 and (5.7)

$$P_n^{(\alpha,-1)}(-1)=0,\quad n\geqslant 1; \tag{5.9}$$

$$P_n^{(\alpha,-1)}(x)'=\frac{n+\alpha}{2}P_{n-1}^{(\alpha+1,0)}(x),\quad n\geqslant 1. \tag{5.10}$$

Then we have by integration by parts,

$$\int_{-1}^{1}P_n^{(\alpha,-1)}(x)(1-x)^{\alpha}\,\mathrm{d}x=\frac{n+\alpha}{2(\alpha+1)}\int_{-1}^{1}P_{n-1}^{(\alpha+1,0)}(x)(1-x)^{\alpha+1}\,\mathrm{d}x=0,\quad n\geqslant 2,$$

since $\{P_n^{(\alpha+1,0)}(x)\}_{n=0}^{\infty}$ is an OPS relative to $(1-x)^{\alpha+1}\mathrm{d}x$ on $(-1,1)$. On the other hand, $\{(1-x)^{\alpha}\mathrm{d}x,(1-x)^{\alpha+1}\mathrm{d}x\}$ is a coherent pair of measures on $(-1,1)$ since we have (see [12])

$$(2n+\alpha+2)P_n^{(\alpha+1,0)}(x)=2P_{n+1}^{(\alpha,0)}(x)'+2P_n^{(\alpha,0)}(x)',\quad n\geqslant 1.$$

Therefore, by Theorem 3.2 and (5.9), $P_n^{(\alpha,-1)}(x)$, $n\geqslant 1$, has n simple zeros in $[-1,1)$ which interlace with the zeros of $P_{n-1}^{(\alpha,0)}(x)$ and $P_{n-1}^{(\alpha+1,0)}(x)$.

Example 3. Jacobi polynomials $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}$ for $\beta>-1$.

For $\beta>-1$, the Jacobi PS $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}$ is orthogonal relative to

$$(p,q)_\lambda^\beta:=\lambda p(1)q(1)+\int_{-1}^{1}p'(x)q'(x)(1+x)^{\beta+1}\,\mathrm{d}x,\quad \lambda\neq 0, \tag{5.11}$$

which is quasi-definite for $\lambda\neq 0$ and positive-definite for $\lambda>0$. Similarly as in Example 2, we can see that $P_n^{(-1,\beta)}$, $n\geqslant 1$, has n simple zeros in $(-1,1]$ which interlace with the zeros of $P_{n-1}^{(0,\beta)}(x)$ and $P_{n-1}^{(0,\beta+1)}(x)$.

Example 4. Jacobi polynomials $\{P_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty}$ for $-(m+1)<\alpha<-m$.

The Jacobi PS $\{P_n^{(\alpha,-1)}(x)\}_{n=0}^{\infty}$ for $\alpha<-1$ and $\alpha\notin\{-1,-2,\ldots\}$ is orthogonal relative to

$$(p,q)_\lambda^\alpha:=\lambda p(-1)q(-1)+\int_{-1}^{1}p'(x)q'(x)(1-x)_+^{\alpha+1}\,\mathrm{d}x,\quad \lambda\neq 0, \tag{5.12}$$

which is quasi-definite. Here the integral $\int_{-1}^{1} \pi(x)(1-x)_+^{\alpha+1}\,dx$ is the regularization of (possibly) divergent integral $\int_{-1}^{1} \pi(x)(1-x)^{\alpha+1}\,dx$ (see [13]). The formulas (5.9) and (5.10) also hold in this case. We now assume $-(m+1) < \alpha < -m$ for some integer $m \geqslant 1$. Then we have by integration by parts

$$\int_{-1}^{1} P_n^{(\alpha,-1)}(x)(1-x)^{\alpha+m}\,dx = \frac{n+\alpha}{2(\alpha+m+1)}\int_{-1}^{1} P_{n-1}^{(\alpha+1,0)}(x)(1-x)^{\alpha+m+1}\,dx$$
$$= 0, \quad n \geqslant m+2,$$

since $\{P_n^{(\alpha+1,0)}(x)\}_{n=0}^{\infty}$ is a TPS relative to $(1-x)_+^{\alpha+1}\,dx$ on $(-1,1)$. On the other hand, $\{(1-x)^{\alpha+m}\,dx, (1-x)^{\alpha+m+1}\,dx\}$ is a coherent pair of measures on $(-1,1)$ since we have (see [12])

$$(2n+\alpha+m+2)P_n^{(\alpha+m+1,0)}(x) = 2P_{n+1}^{(\alpha+m,0)}(x)' + 2P_n^{(\alpha+m,0)}(x)', \quad n \geqslant 1.$$

Therefore, by Theorem 3.3, $P_n^{(\alpha,-1)}(x)$, $n \geqslant m+2$, has at least $n-m-1$ zeros of odd multiplicity in $(-1,1)$. Note also that $x=-1$ is a simple zero of $P_n^{(\alpha,-1)}(x)$, $n \geqslant 1$ (see (5.9) and (5.10)).

Example 5. Jacobi polynomials $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}$ for $-(m+1) < \beta < -m$.

The Jacobi PS $\{P_n^{(-1,\beta)}(x)\}_{n=0}^{\infty}$, $\beta < -1$ and $\beta \notin \{-1,-2,\dots\}$ is orthogonal relative to

$$(p,q)_\lambda^\beta := \lambda p(1)q(1) + \int_{-1}^{1} p'(x)q'(x)(1+x)_+^{\beta+1}\,dx, \quad \lambda \neq 0, \tag{5.13}$$

which is quasi-definite. Similarly as in Example 4, we can see that $P_n^{(-1,\beta)}(x)$, $n \geqslant m+2$, has at least $n-m-1$ zeros of odd multiplicity in $(-1,1)$ when $-(m+1) < \beta < -m$ and $m \geqslant 1$ is an integer.

Example 6. Jacobi polynomials $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$.

When $\alpha = \beta = -1$, the Jacobi differential equation (5.6) has a unique (up to a nonzero constant multiple) polynomial solution of degree n, for every $n \geqslant 0$, $n \neq 1$. For $n=1$, any polynomial $P_1^{(-1,-1)}(x) = x + \gamma$ with arbitrary constant γ is a solution to (5.6). It is shown in [9] that with arbitrary constant γ, $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$ is orthogonal relative to

$$\phi_{A,B}(p,q) := Ap(1)q(1) + Bp(-1)q(-1) + \int_{-1}^{1} p'(x)q'(x)\,dx$$

if A and B are constants such that

$$A+B \neq 0, A(\gamma+1)^2 + B(\gamma-1)^2 + 2 \neq 0 \quad \text{and} \quad A(\gamma+1) + B(\gamma-1) = 0.$$

For example, we may choose $\gamma = 0$. Then $\{P_n^{(-1,-1)}(x)\}_{n=0}^{\infty}$ is orthogonal relative to

$$(p,q)_\lambda := \lambda(p(1)q(1) + p(-1)q(-1)) + \int_{-1}^{1} p'(x)q'(x)\,dx, \lambda \neq 0, -1, \tag{5.14}$$

which is quasi-definite for $\lambda \neq 0, -1$ and positive-definite for $\lambda > 0$. If we set

$$b_{m,n} := mn \int_{-1}^{1} (1+x)^{m+n-2}\,dx, \quad (m \text{ and } n \geqslant 1),$$

then $b_{1,k} = 2^k$ for all $k \geqslant 1$ so that the condition (2.13) is satisfied with $l = 1$. Hence, by Proposition 2.6, we have

$$P_n^{(-1,-1)}(1) = P_n^{(-1,-1)}(-1) = 0, \quad n \geqslant 2; \tag{5.15}$$

$$P_n^{(-1,-1)}(x)' = \frac{1}{2(n-1)} P_{n-1}^{(0,0)}(x), \quad n \geqslant 2. \tag{5.16}$$

Then we have by integration by parts,

$$\int_{-1}^{1} P_n^{(-1,-1)}(x)\,dx = \frac{-1}{2(n-1)} \int_{-1}^{1} x P_{n-1}^{(0,0)}(x)\,dx = 0, \quad n \geqslant 3$$

since $\{P_n^{(0,0)}(x)\}_{n=0}^{\infty}$ is an OPS relative to dx on $(-1, 1)$. On the other hand, $\{dx, dx\}$ is a symmetrically coherent pair of measures on $(-1, 1)$ since we have (see [14])

$$2(n + \tfrac{1}{2})P_n^{(0,0)}(x) = P_{n+1}^{(0,0)}(x)' - P_{n-1}^{(0,0)}(x)', \quad n \geqslant 2.$$

Therefore, by Lemma 4.3 and (5.15), $P_n^{(-1,-1)}(x)$, $n \geqslant 3$, has n simple zeros in $[-1, 1]$. More presicely, if $n = 2m$ is even, then, by Theorem 4.4, $P_{2m}^{(-1,-1)}(x)$ has m positive zeros (resp. m negative zeros) in $(0, 1]$ (resp. in $[-1, 0)$) which interlace with the positive zeros (resp. negative zeros) of $P_{2m-1}^{(0,0)}(x)$ and $P_{2m-2}^{(0,0)}(x)$. If $n = 2m + 1$ is odd, then, by Theorem 4.5, $P_{2m+1}^{(-1,-1)}(x)$ has $2m + 1$ simple zeros in $[-1, 1]$ which interlace with the zeros of $P_{2m}^{(0,0)}(x)$. Also, the positive zeros (resp. negative zeros) of $P_{2m+1}^{(-1,-1)}(x)$ interlace with the positive zeros (resp. negative zeros) of $P_{2m-1}^{(0,0)}(x)$.

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